ELEVEN CONDITIONS ON IDEALS
IN COMMUTATIVE RINGS

BY

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A THESIS

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CHAPTER I

INTRODUCTION

The purpose of this thesis is to compare eleven conditions which may be satisfied by ideals in commutative rings. The conditions, not necessarily listed in order of importance, follow below.

Let $R$ be a commutative ring:

A. $R$ has an identity.

B. $R$ is generated by idempotent elements.

C. If $I$ is a non-zero ideal of $R$ such that $\sqrt{I} \neq R$, then $R/I$ has an identity.

D. $I = RI$ for each ideal $I$ of $R$.

E. If $I$ is a proper ideal of $R$, then $\sqrt{I} \neq R$.

F. $R = R^2$.

G. If $I$ is an ideal of $R$ such that $\sqrt{I}$ is maximal, then $I$ is primary.

H. If $P$ is a non-zero prime ideal of $R$, then $R/P$ has an identity.

J. All maximal ideals are prime.

K. Each proper ideal of $R$ is contained in a maximal ideal.

L. If $X$ and $Y$ are comaximal ideals, $X \cap Y = XY$.

There are two problems which must be taken into consideration. The first is to exhibit the implications which exist among the various conditions; and
the second is to provide as many counterexamples as possible to determine when and if an inequivalence exists. Possibly of lesser importance, but still of some interest, is the problem of discovering what changes occur in the implications if some sort of finiteness condition also holds in the ring. These three problems will be discussed in order in the second, third, and fourth chapters.

Whenever the word ring, or the symbol $R$, is used, the concept of a commutative ring is to be understood. The letter $I$ will always designate ideal. The letters $P$ and $M$ are used to represent prime ideal and maximal ideal, respectively. For typographical convenience, the asterisk, $*$, will serve as the ring addition symbol. All rings are non-zero. Following current usage, $\subseteq$ will denote containment and $\subset$ will represent proper containment.

All definitions are from Zariski and Samuels, (7) and (8). A few necessary definitions which are not normally met in elementary algebra courses are listed.

**DEFINITION 1.** An idempotent element $x$ is an element for which the equation $x^2 = x$ is true.

**DEFINITION 2.** The radical of $I$, $\sqrt{I}$, is the set of all $x$ in $R$, some power of which belongs to $I$.

**DEFINITION 3.** Let $X$ and $Y$ be ideals in $R$. The
product ideal, XY, is the set of all finite sums $\sum x_i y_i$; where $x_i$, $y_i$ are elements of X and Y, respectively.

**DEFINITION 4.** Two ideals, X and Y, are comaximal if they are proper and $X \cdot Y = R$.

**DEFINITION 5.** An ideal M is maximal if $M \neq R$ and if for any ideal Q such that $M \subseteq Q \subseteq R$, then $Q = R$.

**DEFINITION 6.** The ideal $P$ is prime iff whenever the product $ab$ is an element of $P$, then at least one of $a$ or $b$ is an element of $P$.

**DEFINITION 7.** An ideal $Q$ is primary iff whenever the product $ab$ is in $Q$ and $a$ is not, then $b^k$ is in $Q$ for some positive integer $k$.

**DEFINITION 8.** A ring is generated by a set $S$ if an arbitrary element can be written as a finite sum of the form $\sum (n_is_i * r_is_i)$, where the $n_i$ are integers, the $s_i$ are elements of $S$, and the $r_i$ are elements of $R$.

The appendix contains a list of the inequivalences and the counterexamples which furnish them.
CHAPTER II
A CHAIN OF IMPLICATIONS

The title of the chapter explains the purpose of the chapter. Exactly what will be shown is diagrammed below. That these are the only implications which may exist is not claimed, although in chapter 3 enough counterexamples are furnished to show that most of the conditions are necessarily inequivalent. The reader will observe that condition G is absent from the outline, the reason for this being that condition G will be shown to be always true.

A → B → D → F
C → H → L

THEOREM 1. (A implies B) If R has identity, then R is generated by idempotent elements.

Proof. Let S be the set of idempotent elements. Let I be the ideal generated by S. Since the identity, e, of R is idempotent, e is in I. Choose an arbitrary element r of R. It is true that r = re, and re is in I. Thus R ⊆ I. Thus S generates R.

The following two lemmas, while not directly connected with the diagram are of interest and greatly facilitate the proof of the theorem that condition B implies condition D.
LEMMA 1. If $R$ is generated by idempotent elements, then any element $z$ of $R$ can be written as a finite sum 
\[ \sum r_i s_i; \]
where the $s_i$ are idempotent.

Proof. Since $R$ is generated by idempotents $s_i$, we have that if $z$ is in $R$, 
\[ z = \sum (n_i s_i * r_i s_i) = \sum (n_i s_i^2 * r_i s_i) = \sum s_i (n_i s_i * r_i) = \sum s_i r_i; \]
where $r_i = (n_i s_i * r_i)$.

LEMMA 2. If $R$ is generated by idempotent elements, then for any $z$ in $R$ there is an $x$ in $R$ such that $xz = z$.

Proof. Let $R$ be any ring generated by one idempotent $s$. If $z$ is in $R$, then $z = rs$ by the preceding lemma. Now $sz = srs = rss = rs = z$.

Assume that the theorem is true for all rings generated by $k$ elements. Then any $z$ can be represented as a sum 
\[ \sum r_j s_j = w * r_{k+1}s_{k+1}. \]
Since $w$ is in the subring $R'$ generated by $k$ idempotents, there is a $y$ in $R' \subseteq R$ for which $yw = w$. Now $z = w * r's'$, and from the fact that $z(y * s' - ys') = (w * r's')(y * s' - ys')$
\[ = wy * yr's's'w*r's'-s'w-yr's' = z \]
we are finished.

By induction then the theorem is true for all rings since even if there are an uncountable number of generating idempotent elements, each element is generated by only a finite number of elements and hence is contained in a subring generated by a finite number of elements. Further discussion of idempotents is to be found in Jacobson (3, pp. 48-53).
THEOREM 2. (B implies D) If $R$ is generated by idempotent elements, then $I \subseteq RI$ for each ideal $I$ of $R$.

Proof. Let $I$ be any ideal of $R$. Let $x$ be any member of $I$. By lemma 2, there is a $y$ in $R$ such that $xy = x$. Since $xy$ is an element of $RI$, $x$ is in $RI$. Thus $I \subseteq RI$. Obviously $RI \subseteq I$. Consequently, $I = RI$.

THEOREM 3. (D implies E) If $I = RI$ for each ideal $I$ of $R$, then for all proper ideals $I$ of $R$, $\sqrt{I} \neq R$.

Proof. Let $I$ be such that $\sqrt{I} = R$. Let $x$ be any element of $R$. By hypothesis $x = Rx$, where $x$ is the ideal generated by $x$. Thus $x = \sum r_i (n_1 x + t_1 x) = \sum x (r_i n_1 + t_1) = x \sum \bar{r}_1 = xr$. Since $\sqrt{I} = R$, we have that $r^k$ is in $I$ for some positive integer $k$. Since $x = r(rx) = r^k x$ for all $k$, it follows that $x$ is in $I$. As $x$ was arbitrary, we have $R \subseteq I$; and so, $R = I$. Hence if $I$ is proper, $\sqrt{I} \neq R$.

THEOREM 4. (D implies H) $R/P$ has an identity for all non-zero prime ideals $P$ if $I = RI$ for each ideal.

Proof. Let $x$ be an arbitrary element of $R - P$. By hypothesis there is some $r'$ in $R$ such that $x = r's$. Now we wish to show that the coset $(r' + P)$ is the identity in $R/P$. To accomplish this, let $(y + P)$ be a non-zero element in $R/P$. Thus $y$ is not in $P$ and, as above, there is some $s'$ in $R$ such that $ys' = y$. Now $r'xy = xy = s'xy$; so that, $xy(r' - s') = 0$. The product
xy was not in \( P \), hence the factor \((r' - s')\) is in \( P \).

This means that the coset \((r' \ast P)\) is the same as the coset \((s' \ast P)\). It is also true that \((y \ast P)(r' \ast P) = (y \ast P)(s' \ast P) = (ys' \ast P) = (y \ast P)\). Therefore, \((r' \ast P)\) is the identity in \( R/P \).

The next theorem is a generalization of the theorem that in a ring with identity, \( X \cap Y = XY \) for all comaximal ideals \( X, Y \) in \( R \) (7, p. 271)

**Theorem 5.** (D implies L) If \( I = RI \), then \( X \cap Y = XY \) for all comaximal ideals \( X \) and \( Y \) in \( R \).

**Proof.** Let \( X \) and \( Y \) be comaximal ideals of \( R \). Now \( X \cap Y = R(X \cap Y) = (X \ast Y)(X \cap Y) = X(X \cap Y) \ast Y(X \cap Y) \), \( XY \ast XY = XY \). Trivially, \( XY \leq X \cap Y \). Thus \( XY = X \cap Y \).

**Theorem 6.** (D implies F) If \( I = RI \) for all ideals, then \( R = R^2 \).

**Proof.** Immediate.

Kurosh (4, p. 85) proves that in a Dedekind ring all maximal ideals are prime. His method of proof can be generalized to show that if \( R = R^2 \), the conclusion remains valid. This will be done in the following theorem.

**Theorem 8.** (F implies J) If \( R = R^2 \), then all maximal ideals are prime.

**Proof.** Let \( M \) be a maximal ideal in \( R \). Let \( xy \) be a product in \( M \) and \( x \) not be an element of \( M \). If \( y \) is also not an element of \( M \), we have that \( R = (X \ast M) = (Y \ast M) \).
where $X$ and $Y$ are the principal ideals generated by $x$ and $y$. Let $z$ be an arbitrary element of $R$. By hypothesis, $z$ is in $R^2$. However, $R = R^2 = (X * M)(Y * M)$. And so, $z = \sum (n_i x * r_i x * m_i)(n_i y * r_i y * m_i)$. It is easy to verify that in expanding this sum either an $xy, m_i$, or an $m_i$ appears in each summand. Thus $z$ is in $M$. But this contradicts the fact that $M \not= R$. Consequently, if $x$ is not in $M$, $y$ is in $M$; that is, $M$ is prime.

**Theorem 8.** (D implies K) If $I = RI$ for all ideals in $R$, then each proper ideal is contained in a maximal ideal.

**Proof.** Let $I$ be any proper ideal in $R$. By theorem 3, $\forall I \not= R$. Since $\forall I = \bigcap P_i$ where the $P_i$ are the minimal prime ideals for which $I \subseteq P_i$ (5, p. 104), we have that $I$ is in a proper prime ideal $P$. By theorem 4, $R/P$ has an identity; and so, $P/P$ is in some maximal ideal $M/P$.

From the one-one correspondence between ideals in $R$ which contain $P$ and the ideals in $R/P$, we have that $I \subseteq P \subseteq M'$ where $M'$ is a maximal ideal corresponding to $M/P$.

**Theorem 9.** (A implies C) If $R$ has an identity, $R/I$ has an identity if $I$ is such that $\forall I = R$.

**Proof.** By hypothesis the identity of $R$, $e$, is not an element of $I$. It follows directly that the coset $(e * I)$ is the identity in $R/I$ since $(e * I)(x * I) = (ex * I) = (x * I)$ for all $(x * I)$ in $R/I$. 
THEOREM 10. (C implies H) If $R/I$ has an identity for all ideals $I$ such that $\sqrt{I} \neq R$, then $R/P$ has an identity if $P$ is a non-zero prime ideal.

Proof. If $P$ is prime, then $\sqrt{P} = P$ since $r^k$ is an element of $P$ iff $r$ is an element of $P$. Thus if $P \neq R$, $\sqrt{P} \neq R$. Consequently, $R/P$ has an identity if $P$ is proper. If $P = R$, $R/P = 0$ and the theorem is trivially true.

We will close this section with a proof of the theorem that condition $G$ is always true. From this it will be obvious that all conditions then imply $G$; and this explains why condition was nowhere to be found in the diagram given at the beginning of this chapter.

THEOREM 11. If an ideal of $R$ is such that $\sqrt{I}$ is maximal, then $I$ is primary.

Proof. Let $\sqrt{I}$ be maximal. Let $bc$ be an element of $I$. Suppose $b$ is not in $I$, then $c = a * nb * rb$ where $a$ is in $\sqrt{I}$, $a^k$ is in $I$ for some positive integer $k$. Now $c^2 = ac * nbc * rbc = ac * c'$ where $c' = nbc * rbc$ is in $I$.

It follows then that $c^{2k} = (ac * c')^k = a^{ek} * \sum (a, b, c, r) \cdot c'^r$. Thus $c^{2k}$ is in $I$; and so, $I$ is primary.
CHAPTER III
COUNTEREXAMPLES

The primary concern of this chapter is to determine as many inequivalences as possible. The counterexamples have not been chosen for any reason other than their availability. At times there may be a duplication in the sense that although it would be sufficient to give an example in which condition C occurred but not B to show that B and C are inequivalent, it might be that an example of B but not C will also be given to show that the outline of implications is reasonably complete.

No attempt has been made to state all inequivalences demonstrated by each counterexample. From the chain of implications given at the beginning of chapter 2, it is obvious that the first counterexample, in which condition K holds although condition J does not, will also serve to show that K but not F, K but not D, K but not B, and K but not A are implied. However, only the statement that the counterexample yields K but not J will be given. With the appendix and the previously mentioned diagram, the reader can decide for himself what has been shown in each counterexample.

The ring of even integers, subsequently denoted by E, furnishes a wealth of inequivalences once the nature of the ideals in E has been examined. The most important
fact to be established is that if $I$ is an ideal of $E$, then $I$ is also an ideal of $Z$, the integers.

To prove the last statement, we will let $I$ be any ideal of $E$. Since $I$ is closed under addition in $E$, it is certainly closed in $Z$. Multiples of any element of $I$ by an even integer are in $I$; so that all that remains to be examined is multiplication of an element of $I$ by an odd integer. Let $k$ be odd, then $k = 2h + 1$. Let $2s$ be any element of $I$. Then $2s(2h + 1) = 2s2h + 2s$. As both $2s2h$ and $2s$ are in $I$, so is $2sk$ in $I$. Thus $I$ is an ideal in $Z$.

**Counterexample 1.** Since all ideals in $Z$ are principal, each proper ideal of $E$ consists of the multiples of some element $2p_1^{e_1} \cdots p_n^{e_n}$, where the $p_i$ are prime integers. Now if $I$ is generated by $2p_1^{e_1} \cdots p_n^{e_n}$, then $I$ is obviously contained in the ideal generated by $2p_1$, $(2p_1)$. To show that $(2p)$ is maximal we let $(2p) \subseteq (2h)$. By the nature of the ideals in $Z$, there must be an $x$ in $Z$ such that $2hx = 2p$ or $hx = p$. Since $p$ is prime, either $h = p$ or $h = 1$. Thus $(2h) = E$ or $(2h) = (2p)$. Consequently, every ideal in $E$, $(2m)$, is contained in $(2p)$ where $p$ is a prime integer appearing in the unique factorization of $m$, and this fact implies that condition $K$ is true in $E$.

From the last paragraph we have that $(4)$ is maximal, yet $2 \cdot 2$ is in $(4)$ although $2$ is not. Therefore, not all
maximal ideals are prime. Thus condition $J$ does not hold in $E$.

Next we will examine condition $L$ in the ring $E$. If we allow $X$ to be the ideal (4), $Y$ to be the ideal (6), then $X$ and $Y$ are comaximal since $2 = 6 - 4$ is in $X \cap Y$; and so, $X \cap Y = R$.

If $r$ is in $X \cap Y$, then $r = 4x = 6y$ or $2x = 3y$ for certain integers $x$ and $y$. Obviously $x$ is divisible by 3 so that $x = 3k$. Hence $4x = 4(3k) = 12k$; and so, $r = 12k$ which is in (12). On the other hand if $12h$ is in (12), then $12h = 4(3h) = 6(2h)$; and so, $12h$ is in $X \cap Y$ Thus $X \cap Y = (12)$.

In like manner it can be shown that $(24) = XY$.

Proof of this rests on the fact that if $z$ is in $XY$, then $z = \sum 4x_1y_1 = \sum 24x_1y_1$, so that $z$ is in (24). If $24n$ is in (24) then $24n = 6 \cdot 4n$, and hence is in $XY$. Therefore, $(24) = XY \neq X \cap Y = (12)$

We can also derive that condition $C$ is not valid in the ring $E$. This can be accomplished easily by examining the ideal generated by 12, (12). Since (12) is an ideal in $Z$, we have that if a non-zero $2^k$ is in (12), then $2^k = 12z$ for some integer $z$. Thus $2^{k+2} = 3 \cdot 2^k$ which means that 3 divides $2^k$ which is impossible if $2^k \neq 0$.

Hence, $\sqrt{(12)} \neq E$. However, since in $E/(12)$ we have $\overline{12} = \overline{0}$, $\overline{2} = \overline{4}$, $\overline{24} = \overline{3}$, $\overline{82} = \overline{4}$, and $\overline{16 \cdot 2} = \overline{5}$, there is no iden-
ity for $E$, and thus there is no identity in $E/(12)$.

To show that condition H is valid in $E$ we must examine the prime ideals in $E$. We will prove that any prime ideal $P$ in $E$ is of the form $(2p)$ where $p$ is some prime integer distinct from 2.

Part of this has already been shown, in particular that (4) is not prime. Now let $I$ be an ideal generated by $2m$ where $m = p_1^{e_1} \cdots p_n^{e_n}$. Obviously the product $2p_1 \cdot 2p_1^{-1} \cdots p_n$ is in $(2m)$ although neither of the factors separated by · is in $(2m)$. Hence if $m$ has more than one prime integer appearing in its unique factorization, or if a power higher than the first of any prime appears, then the ideal $(2m)$ is not prime.

However, if $I = (2p)$, $p$ defined as above, and if $2x \cdot 2y$ is in $(2p)$, then $2x2y = 2ph$; and so, $p$ divides the product $xy$. Hence $p$ divides $x$ or $p$ divides $y$. Thus either $2x$ or $2y$ is in $(2p)$, and $(2p)$ is prime.

In any coset ring $E/I$ we have that $2n = 2\bar{n}$. Thus if an identity can be obtained for $2$ so that $\bar{e2} = \bar{2}$, then $e2n = en\bar{2} = n\bar{e2} = n\bar{2} = \bar{2n}$.

Let $P$ be a prime ideal, $P = (2p)$. From the preceding discussion we know that $p$ must be an odd integer and $p \cdot 1$ is an even integer. We wish to show that the coset $\bar{p \cdot 1}$ is the identity in $E/(2p)$. From the product $2(\bar{p \cdot 1}) = 2p \cdot 2$, we see that $2p \cdot 2$ and 2 are in the
same coset since \((2p * 2) - 2 = 2p\) which is in \((2p)\).

Therefore, \(p * 1\) is the identity in \(E/(2p)\).

**COUNTEREXAMPLE 2.** In order to get a ring in which condition \(L\) holds we must go to a slightly less familiar ring, the direct sum of \(E/(4) \oplus E/(4) = \{(0,0),(0,2),(2,0),(2,2)\}\) with componentwise operations following the rules \(2 * 2 = 0\) and \(22 = 0\). A moment's inspection verifies that \(X = \{(0,0),(2,2)\}\), \(Y = \{(0,0),(2,0)\}\), and \(W = \{(0,0),(2,2)\}\) are the only proper ideals. Furthermore, any combination of two of them are certainly co-maximal. Closer scrutiny yields that \(X \cap Y = X \cap W = Y \cap W = (0,0)\). Thus \(L\) holds in \(R\).

However, although the ideal \(X\) is obviously maximal, it is not prime since \((2,0)(2,0) = (0,0)\) is in \(X\) whereas \((2,0)\) is not in \(X\). Thus \(J\) does not hold.

**COUNTEREXAMPLE 3.** Let \(R\) be the ring \(E/(12)\). Since there is a one-one correspondence between ideals in \(E/(12)\) and ideals in \(E\) which contain \((12)\), there are only two ideals in \(R = \{0,2,4,6,8,10\}\). One of the ideals is \(X = \{0,4,8\}\) and the other is \(Y = \{0,6\}\). Both are maximal. It is obvious that \(\sqrt{X} = R\) since \(2^2 = 4\), and it is equally obvious that \(\sqrt{Y} \neq R\) since \(2^k\) is not in \(Y\) for any \(k\). In \(R/Y = \{0 * \{0,6\}, 2 * \{0,6\}, 4 * \{0,6\}\}\), it is true that the element \(4 * \{0,6\}\) is the identity. Thus condition \(C\) is true. However, not all maximal ideals are prime since
\( \overline{e} \cdot \overline{e} = \overline{e} \) is in \( X \) whereas \( \overline{e} \) is not. \( J \) does not hold.

**Counterexample 4.** That condition \( B \) may be valid although condition \( A \) is not is the essence of this example.

If \( S \) is any infinite set, let \( R \) be the set of all finite subsets of \( S \). Under the definitions \( X \cdot Y = X \cup Y \) \(-\) \( X \cap Y \) and \( XY = X \cap Y \) for subsets of \( S \), \( R \) is a ring. Let the set \( Q \) be any element of \( R \). Since \( Q \) is finite, there is some singleton set \( x \) in \( S \) such that \( x \) is not contained in \( Q \). Let \( Q' = Q \cup x \). Then \( Q'Q = Q \neq Q' \), so that \( Q \) is not the identity of \( R \). As \( Q \) was completely arbitrary, there is no identity in \( R \). However, every element in \( R \) is idempotent, and this implies that \( R \) is generated by idempotent elements.

Further study will show that condition \( C \) is not true in the ring. We must begin by considering an ideal generated by a singleton set \( x \), and this ideal shall be designated by \( X \). Any element of \( X \) is of the form \( nx \cdot Yx \) by definition. But \( nx = \emptyset \) or \( nx = x \), and \( Yx \) yields the same answer. Thus \( nx \cdot Yx \) is either of the two sets \( \emptyset \) or \( x \). Since \( Y^k \) is in \( X \) iff \( Y \) is in \( X \), \( \sqrt{X} \neq R \). Let \( (W \cdot X) \) be a coset in \( R/X \). Let \( (T \cdot X) \) be the coset where \( T = W \cup z \) and \( z \) is a singleton set not in \( W \). Now \( (T \cdot X) = (TW \cdot X) = (W \cdot X) \). Obviously \( (T \cdot X) \neq (W \cdot X) \). And since \( W \) was arbitrary, there is no identity in \( R/X \).
**Counterexample 5.** (K but not H) Let \( R \) be the direct sum, \( E \times F \), of the set of even integers and any field. By the isomorphism theorems, \( R/F \) is isomorphic to \( E \), and thus \( R/F \) does not have an identity. However, \( 0 \times F \) is prime since if \((e_1,f_1)(e_2,f_2) = (0,f_3)\), then \(e_1e_2 = 0\). Therefore, one of the factors must be zero, and so, one of the factors is in \( 0 \times F \). Hence condition H does not hold.

Let \( I \) be an ideal in \( R \). Let \( X = \{ x: x \text{ is in } E \text{ and } (x,f) \text{ is in } I \} \). For any \( y \) in \( E \) and \( x \) in \( X \) we have that \((y,g)(x,f) \text{ is in } I \). Also if \( x \) and \( y \) are in \( X \), then \((x,f) \times (y,g) \text{ are in } I \). Consequently, \( X \) is an ideal in \( E \). If we let \( Y = \{ f: (x,f) \text{ is in } I \} \), by symmetry \( Y \) is an ideal of \( F \). Thus the ideals of \( R \) are contained in ideals of the form \( X \times Y \). On the other hand, if \( M \) is a maximal ideal of \( E \), then \( M \times F \) is certainly a maximal ideal of \( R \) because \((a,f) \text{ is not an element of } M \times F \text{ iff } a \text{ is not an element of } M \). In summary, every ideal \( I \) is contained in a maximal ideal \( M \times F \). Therefore, condition K holds.

**Counterexample 6.** (E but not F,D) Let \( R \) be any cyclic group of prime order in which multiplication is defined trivially, \( xy = 0 \) for all \( x,y \) in \( R \). Condition E is satisfied since there are no proper ideals in \( R \), yet \( R = R^2 \); and so, \( F \) does not hold. Obviously \( A \neq RA \) for each ideal; and so, condition D also fails.
COUNTEREXAMPLE 7. (L but not E). Let $G$ be the non-cyclic group of four elements, $G = \{0, x, y, x \ast y\}$. Define multiplication trivially. $G$ is then a ring with three proper subgroups: 1) $X = \{0, x\}$, 2) $Y = \{0, y\}$, 3) $Z = \{0, x \ast y\}$. And because of the definition of multiplication $X = Y = Z = G$. Therefore condition E does not hold.

However, any combination of two of $X, Y, \text{or} Z$ is a comaximal pair, and $X \cap Y = X \cap Z = Y \cap Z = XY = XZ = YZ = 0$; and so, condition L is true.

Only three counterexamples are now necessary to establish that the eleven conditions are at least inequivalent; 1) an example in which J is true but not F, 2) an example in which F occurs but not D, 3) a ring for which D is valid but not B. That the author of this thesis has not been able to produce such examples is, as implied in the preceding sentence, true. However, the following three rings, with due credit to their source, will be listed as possibilities. It must be understood that no claim is made that they are indeed suitable for the purpose.

Mr. Robert W. Gilmer, the original formulator of the problem, maintains that if $R$ is the additive group of rationals, then J holds whereas F does not.

The same author (2, pp. 237-250) states that if $Q$ is a valuation ring whose maximal ideal is the union of
the prime ideals properly contained in $M$ and $Q = k \ast M$; where $k = GF(p)$ for some prime $p$, then $F$ but not $D$.

Let $k$ be a field and $\{X_i\}_{i=1}^{\infty}$ be a set of algebraically independent elements from an extension field over $k$. Let $M$ be the ideal of $k(\{X_i\})$ generated by the $\{X_i\}_{i=1}^{\infty}$ and $N$ be the ideal generated by $\{X_i - X_iX_j\}_{i < j}$. Gilmer says that if $R = M/N$, $D$ holds but not $B$. This example comes from a work of Gilmer, "A counterexample to a conjecture of Krull," submitted for publication and to which access has not been made available.
CHAPTER IV

FINITENESS CONDITIONS

In this chapter we consider the relationships among a few of the eleven conditions when additional stipulations are placed upon the ring itself. In particular we are interested in what happens if the maximal condition, or the finite basis or ascending chain condition (1. pp. 113-114), hold in the ring.

The first requirement we will impose upon the ring is that multiplication is not defined trivially. In a ring of this type, we can prove that condition E implies condition F, irregardless of whether the maximal condition holds or not.

THEOREM 12. If multiplication is not defined trivially and if $\sqrt{I} \not\subseteq R$ for all proper ideals $I$ of $R$, then $R = R^2$.

Proof. Consider the ideal $R^2$. We know that $\sqrt{R^2} = R$ since if $r$ is an arbitrary element of $R$, $r^2$ is in $R^2$. Thus $R^2$ is not proper. Hence $R^2 = 0$ or $R^2 = R$. Since $R^2 = 0$ iff $xy = 0$ for all products, contrary to the hypothesis, $R = R^2$.

From this point the maximal hypothesis is assumed for all rings although in lemma 3 no use is made of the fact. We will close this chapter with one more proposition in addition to lemma 3. Let $R$ be generated by $n$ elements $\{x_i\}_{i=1}^n$. 
and for each \( x_i \) let there be an \( r_i \) in \( R \) for which \( r_i x_i = x_i \), then \( R \) has an identity.

Proof. Let \( R \) be any ring generated by one such \( x' \).

If \( w \) is in \( R \), \( w = nx' * sx' \). If \( r'x' = x' \), \( r'w = r'nx' * r'sx' = nx' * sx' = w \).

Assume that the theorem is true for all rings generated by \( n < k \) such elements. Let \( R \) be generated by \( \{x_i\}_{i=1}^{k} \), and for each \( x_i \) there is an \( r_i \) as required. In the ring \( R' \), generated by the first \( k-1 \) elements, by the inductive hypothesis, there is an identity in \( R' \), \( e \), for which \( ew = w \) for all \( w \) in \( R' \). Now let \( z \) be any element of \( R \). Writing \( x_k \) as \( x' \), we have that \( z = w * r'x' \) where \( w \) is in \( R' \). Now let \( y \) be the element in \( R \) for which \( x' = yx' \), then \( (e * y - ey)(w * r'x') = w * wy - wy * er'x' * r'x' - er'x' = w * r'x' = z \). Thus \( (e * y - ey) \) is the identity in \( R \).

**Theorem 13.** (D implies A) If \( I = R I \) for all ideals, then \( R \) has an identity.

Proof. Since the maximal condition holds, \( R \) is generated by a finite number of elements. For any element, and in particular for any generating element \( x \), \( (x) = Rx \), and hence there is an \( r' \) in \( R \) for which \( x' = r'x' \). By the previous lemma, \( R \) has an identity.

We will close this chapter with a theorem from which, in addition to the other theorems preceding it, we can
see that in a ring satisfying the maximal hypothesis, conditions A, B, D, F, J are equivalent.

THEOREM 14. (J implies F) If all maximal ideals of R are prime, $R = R^2$.

Proof. Let $R^2 \leq X \subset R$. If $r$ is any element of R, $r^2$ is in $R^2 \subseteq X$. Therefore X is not prime; and so, not maximal. Now if $R \not\subset R^2$, then $R^2 \subseteq M \subset R$. Hence $R^2 = R$. 
Following each pair, one of which is true and the other is false (indicated by $\times$), is the number of the counterexample in which the inequivalence exists; that is, $(\mathcal{A}, \mathcal{B} = 4)$ indicates that in counterexample 4, $\mathcal{A}$ is false whereas $\mathcal{B}$ is true. The $'$ following an expression indicates that the inequivalence is true if Gilmer's counterexamples are valid.

\[ \begin{align*}
\mathcal{A}, \mathcal{B} = 4 & \quad \mathcal{E}, \mathcal{J} = \times \quad \mathcal{F}, \mathcal{K} = 1 \\
\mathcal{A}, \mathcal{C} = 3 & \quad \mathcal{E}, \mathcal{K} = 1 \\
\mathcal{A}, \mathcal{D} = 4 & \quad \mathcal{E}, \mathcal{L} = 2 \\
\mathcal{A}, \mathcal{E} = 4 & \quad \mathcal{C}, \mathcal{E} = 3 \\
\mathcal{A}, \mathcal{F} = 4 & \quad \mathcal{C}, \mathcal{E} = 3 \\
\mathcal{A}, \mathcal{G} = 1 & \quad \mathcal{C}, \mathcal{F} = 3 \\
\mathcal{A}, \mathcal{H} = 3 & \quad \mathcal{G}, \mathcal{K} = 1 \\
\mathcal{A}, \mathcal{J} = 4 & \quad \mathcal{G}, \mathcal{K} = 1 \\
\mathcal{A}, \mathcal{K} = 1 & \quad \mathcal{K}, \mathcal{L} = 1 \\
\mathcal{A}, \mathcal{L} = 2 & \quad \mathcal{K}, \mathcal{L} = 1 \\
\mathcal{B}, \mathcal{C} = 3 & \quad \mathcal{F}, \mathcal{K} = 1 \\
\mathcal{B}, \mathcal{D} = 4 & \quad \mathcal{F}, \mathcal{L} = 4 \\
\mathcal{B}, \mathcal{E} = 6 & \quad \mathcal{F}, \mathcal{L} = 6 \\
\mathcal{B}, \mathcal{F} = \times & \quad \mathcal{F}, \mathcal{L} = 2 \\
\mathcal{B}, \mathcal{G} = 1 & \quad \mathcal{G}, \mathcal{L} = 5 \\
\mathcal{B}, \mathcal{H} = 3 & \quad \mathcal{G}, \mathcal{L} = 3 \\
\mathcal{B}, \mathcal{J} = \times & \quad \mathcal{G}, \mathcal{K} = 1 \\
\end{align*} \]
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